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On variable viscosity magma flow

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Abstract—The flow of a fluid in a pipe was treated by Poiseuille and one may find in Landau and Lifshitz a generalization of Poiseuille's solution for a fluid the temperature of which varies in a section. Platten and Legros treat a flow with a variable viscosity, but the expression for the viscosity they use, does not describe the viscosity of a liquid. The solution we propose here concerns the flow of magma, the viscosity of which is a rapidly changing function of temperature. The example we calculate numerically is the flow of the basaltic magma, obeying the viscosity law of liquids and its characteristics are listed in a paper by Hardee.

1. INTRODUCTION

THE FLOW of a fluid in a pipe was treated by Poiseuille and one may find in Landau and Lifshitz [1] a generalization of Poiseuille's solution for a fluid the temperature of which varies in a section. Platten and Legros [2] treat a flow with a variable viscosity, but the expression for the viscosity they use does not describe the viscosity of a liquid.

The solution we propose here concerns the flow of magma, the viscosity of which is a rapidly changing function of temperature (Fig. 1). The example we calculate numerically is the flow of basaltic magma, obeying the viscosity law of liquids [3] the characteristics of which are listed in Hardee [4].

2. EQUATIONS

We intend to treat a stationary flow along the axis of a cylindrical pipe. The temperature of the pipe wall is maintained constant. The viscosity of the flowing fluid depends on its temperature and obeys the law

$$\mu = \mu_0 \, e^{A/kT} \qquad (1)$$

where k represents the Boltzmann constant ($k = 1.3803 \times 10^{-23}$ J K^{-1} mol^{-1}).

The fundamental equations of the steady flow, for a variable viscosity fluid, are given below.

The continuity equation

$$\nabla_1(\rho v) = 0. \qquad (2)$$

The momentum equation

$$\rho v \nabla_0 v + \nabla_0 p = [\nabla_0 v + (\nabla_0 v)^T] \cdot \nabla_0 \mu + \mu \nabla_0 (\nabla_1 \cdot v) + \mu \nabla_1 \cdot (\nabla_0 v) + \nabla_0[(\xi - \tfrac{2}{3}\mu)(\nabla_1 \cdot v)] \qquad (3)$$

where ∇_1 represents the divergence operator

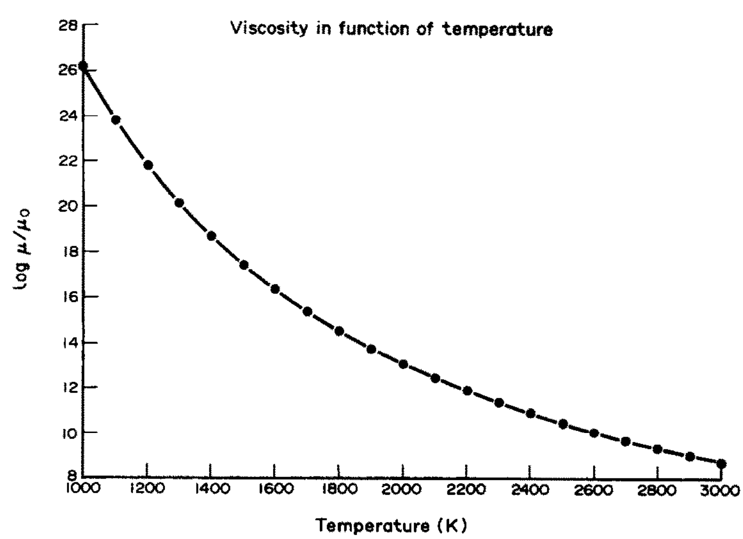

FIG. 1. Viscosity of the magma as a function of temperature.

NOMENCLATURE

A	constant
D	viscous dissipation
e	internal energy
p	pressure
q	heat
Q	mass flow rate
r	radial variable
R	radius of the conduit
T	temperature of the fluid
v	velocity of the fluid.

Greek symbols

λ	heat conductivity
μ	viscosity of the fluid
ξ	bulk viscosity
ρ	density of the fluid
ϕ	shear reported to the length.

Other symbols

∇_1	divergence operator
∇_0	gradient operator.

$$\left(\frac{1}{r}\frac{\partial}{\partial r},\ \frac{1}{r}\frac{\partial}{\partial \vartheta},\ \frac{\partial}{\partial z}\right)$$

∇_0 represents the gradients operator

$$\left(\frac{\partial}{\partial r},\ \frac{1}{r}\frac{\partial}{\partial \vartheta},\ \frac{\partial}{\partial z}\right)$$

and r, ϑ and z represent the radius, the polar angle and the axial coordinate, respectively.

The energy equation

$$\rho v\cdot\nabla_0\left(\frac{v^2}{2}+e\right)+\nabla_1 q+\nabla_1(vp)$$
$$=\mu\nabla_1\left[v\cdot\nabla_0 v+\nabla_0\frac{v^2}{2}\right]+\left[v\cdot\nabla_0 v+\nabla_0\frac{v^2}{2}\right]\cdot\nabla_0\mu$$
$$+\nabla_1\cdot[(\xi-\tfrac{2}{3}\mu)v(\nabla_1\cdot v)]. \quad (4)$$

The heat equation

$$q=-\lambda\nabla_0 T \quad (5)$$

where λ is the heat conductivity.

3. NON-DIMENSIONAL EQUATIONS

In what follows, we put these equations, containing variable viscosity terms, into a non-dimensional form. To do that, we use as reference values the pipe radius R, the wall temperature T_0, the viscosity μ_0 taken at the wall temperature, the pressure p_1 at the entry of the pipe, the density ρ_1 of the fluid at the entry, and the mean velocity of flow v_m, which introduces also a mass flow rate at the entry

$$Q_1=\rho_1 v_m \pi R^2. \quad (6)$$

In fact, as the wall temperature remains uniform, we also have $T_1 = T_0$, and $\mu_1=\mu_0$. In that way all the reference values are taken at the entry of the pipe.

The similarity parameters which appear in the non-dimensionalizing operation are

$$Re=\frac{\rho_0 v_m R}{\mu_0}$$

$$Eu=\frac{p_0}{\rho_0 v_m^2}$$

$$Pr=\frac{c_v\mu_0}{\lambda}$$

$$Br=\frac{\mu_0 v_m}{\lambda T_0}.$$

The heat conductivity λ is supposed constant, as well as the specific heat c_v. The density ρ_0 is also assumed constant. This assumption limits the application of the theory to the flow of liquids, the thermal expansion coefficients of which are very low (e.g. the thermal expansion coefficient of liquid volcanic magma is of the order of 10^{-4}–10^{5} K^{-1}, and the coefficient for liquid glass is of the order of 10^{-5}–10^{-6} K^{-1}).

Introducing these parameters and applying the internal energy relation

$$\mathrm{d}e=c_v\,\mathrm{d}T+\frac{p}{\rho^2}\mathrm{d}\rho-\frac{T}{\rho^2}\left(\frac{\partial p}{\partial T}\right)\mathrm{d}\rho \quad (7)$$

the terms containing $\mathrm{d}\rho$ disappear, as grad $\rho=0$ for $\rho=$ const. and we get the following system of non-dimensional equations (we use here for the non-dimensional values the same symbols used precedently for dimensional ones):

$$\nabla_1(\rho v)=0 \quad (8)$$

$$Re\,(\rho v\cdot\nabla_0 v+Eu\,\nabla_0 p)=[\nabla_0 v+(\nabla_0 v^T)]\cdot\nabla_0\mu$$
$$+\mu[\nabla_0(\nabla_1\cdot v)+\nabla_1\cdot(\nabla_0 v)]$$
$$+\nabla_0[(\xi-\tfrac{2}{3}\mu)(\nabla_1\cdot v)] \quad (9)$$

and

$$Re\,Pr\,Br^{-1}(\rho v\cdot\nabla_0 T)-Br^{-1}\nabla_1\cdot\nabla_0 T$$
$$=(\xi-\tfrac{2}{3}\mu)(\nabla_1\cdot v)^2+\mu\{[\nabla_0 v+(\nabla_0 v^T)]\cdot\nabla_0\}\cdot v. \quad (10)$$

For a steady flow along the axis of a circular pipe, these equations may be written as

$$\partial/\partial z(\rho v)=0. \quad (11)$$

For the radial component of the momentum equation

$$Re\,Eu\frac{\partial p}{\partial r}=\frac{\partial}{\partial r}\left[(\xi-\tfrac{2}{3}\mu)\frac{\partial v}{\partial z}\right]+\frac{\partial}{\partial z}\left[\left(\mu\frac{\partial v}{\partial r}\right)\right]. \qquad (12)$$

For the axial component of the momentum equation

$$Re\,\rho v\frac{\partial v}{\partial z}+Re\,Eu\frac{\partial p}{\partial z}=\frac{\partial}{\partial z}\left[(\xi+\tfrac{4}{3}\mu)\frac{\partial v}{\partial z}\right]+\frac{1}{r}\frac{\partial}{\partial r}\left(r\mu\frac{\partial v}{\partial r}\right). \qquad (13)$$

For the energy equation

$$Re\,Br^{-1}\,Pr\,\rho v\frac{\partial T}{\partial z}-Br^{-1}\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial T}{\partial r}-Br^{-1}r^{-1}\frac{\partial^2 T}{\partial z^2}=(\xi+\tfrac{4}{3}\mu)\left(\frac{\partial v}{\partial z}\right)^2+\mu\left(\frac{\partial v}{\partial r}\right)^2. \qquad (14)$$

Equations (11)–(14) will be applied to solve the problem, with the boundary conditions (in non-dimensional form), as

$$\text{for } r=1,\quad T=1,\quad v=0 \qquad (15)$$

which means that the temperature of the wall does not change, and that the velocity of the fluid in contact with the pipe wall is zero.

Other conditions are

$$\text{for } r=0,\quad \partial T/\partial r=0 \quad\text{and}\quad \partial v/\partial r=0. \qquad (16)$$

These say, that the axis of the tube ($r=0$) represents also the axis of symmetry for the distribution of temperature and of velocity in the tube.

4. ANALYSIS OF THE EQUATIONS

Assuming that the global heat generation by viscous effect is continuously annulled on the wall, the problem is stationary, and the temperature of the pipe wall is uniform. One has

$$\partial T/\partial z=0 \qquad (17)$$

and, from equation (11) it follows that

$$\rho v=f(r) \qquad (18)$$

where from

$$\frac{\partial v}{\partial z}=-\frac{f(r)}{\rho^2}\frac{\partial \rho}{\partial z}. \qquad (19)$$

The non-dimensional mass flow rate is written as

$$1=2\int_0^1 f(r)r\,\mathrm{d}r. \qquad (20)$$

For some fluids, the temperature dependence of the density is very slight; we suppose here that the volcanic magma does not change its volume notably under variable temperature conditions; the theory applies for constant density flows.

From equations (18) and (19) we have

$$\frac{\partial v}{\partial z}=-v\left(\frac{1}{\rho}\frac{\partial \rho}{\partial T}\right)\frac{\partial T}{\partial z} \qquad (21)$$

and, by equation (17)

$$\partial v/\partial z=0. \qquad (22)$$

We may now write the components of the momentum equations (12) and (13) as

$$Re\,Eu\frac{\partial p}{\partial r}=0 \qquad (23)$$

and

$$Re\,Eu\frac{\partial p}{\partial z}=\frac{1}{r}\frac{\partial}{\partial r}\left(r\mu\frac{\partial v}{\partial r}\right) \qquad (24)$$

and the equation of energy (14) takes the form

$$\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial T}{\partial r}=\mu\left(\frac{\partial v}{\partial r}\right)^2 Br. \qquad (25)$$

The viscosity μ depending on temperature only, gives

$$\partial p/\partial z=\text{const.} \qquad (26)$$

This comes from the fact, that, after equation (23) we have $\partial p/\partial r=0$, and that leaves us with $p=p(z)$; but after equations (17) and (22), the right-hand side of equation (24) is independent of z, so equation (26) must be true. In consequence, we also have

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\mu\frac{\partial v}{\partial r}\right)=\text{const.} \qquad (27)$$

Introducing the notation

$$2\phi=Re\,Eu\frac{\partial p}{\partial z} \qquad (28)$$

we have, integrating equation (27)

$$\mu\frac{\partial v}{\partial r}=\phi r. \qquad (29)$$

This relation may, in fact, be found in Landau and Lifshitz [1]. The difference consists in the fact, that the viscosity μ is here a function of temperature.

Introducing relation (29) into equation (25), we have

$$\frac{1}{r^3}\frac{\mathrm{d}}{\mathrm{d}r}r\frac{\mathrm{d}T}{\mathrm{d}r}=-\phi^2\,Br\frac{1}{\mu}. \qquad (30)$$

Relation (29) proves, that there exists only one extremum for the velocity v, and that it is situated in the centre of the pipe, for $r=0$. So, the function which represents v is strictly monotonous, and, as $v=0$ on the wall, it must be a strictly decreasing function, μ having a finite value and ϕ a constant. The same reasoning proves that ϕ is negative.

Equations (29) and (30) involve the velocity and the temperature through the relation

$$\frac{dv}{dr} = -\frac{1}{Br\,\phi}\frac{1}{r^2}\frac{d}{dr}r\frac{dT}{dr}. \quad (31)$$

This relation means also, that for $r = 0$, the limit of

$$\left(\frac{1}{r^2}\frac{d}{dr}r\frac{dT}{dr}\right) = 0:$$

the value of the velocity on the wall is zero, as stated precedently.

5. METHOD OF RESOLUTION

Equation (30) with boundary conditions (15) and (16) does not seem to have a general analytical solution. For very simplified conditions of flow ($\mu = \text{const.}$) one obtains the Poiseuille solution; Platten and Legros treat a problem, where the viscosity is represented by

$$\mu = e^{-n(T-1)}.$$

The solution we present here may be used for different expressions of viscosity.

In the present paper we shall apply, for the viscosity, the expression [3]

$$\mu = \mu_0\, e^{A/kT}.$$

This is a general expression for the viscosity of liquids; we shall apply it here for basaltic magma flows, using the physical parameters which may be found in the paper by Hardee [4].

The solution of equation (30) we propose here, satisfies this equation at the boundary (situated on the wall of the tube) and at the centre of the flow. Besides, this solution satisfies the differential of the equation as well at the boundary (on the wall) as in the centre of the flow.

So, we impose the following conditions to be satisfied:

$$\text{for } r = 1, \quad T = 1$$
$$\text{for } r = 0, \quad dT/dr = 0. \quad (32)$$

For $r = 1$ we have (from equation (30))

$$\frac{d}{dr}\left(r\frac{dT}{dr}\right) = -\phi^2\,Br. \quad (33)$$

For $r = 0$

$$\lim_{r\to 0}\frac{1}{r^3}\frac{d}{dr}\left(r\frac{dT}{dr}\right) = -\phi^2\,Br\frac{1}{\mu[T(0)]}. \quad (34)$$

For the differential of equation (30) on the wall ($r = 1$), we have

$$-3\left(\frac{d}{dr}r\frac{dT}{dr}\right) + \frac{d^2}{dr^2}\left(r\frac{dT}{dr}\right) = \phi^2\,Br\frac{d\mu}{dT}\frac{dT}{dr}. \quad (35)$$

At the centre of the tube, we have

$$\lim_{r\to 0} -3\left(\frac{d}{dr}r\frac{dT}{dr}\right) + \frac{d^2}{dr^2}\left(r\frac{dT}{dr}\right) = 0. \quad (36)$$

To solve equation (30) we use the power series development

$$\sum a_n r^n. \quad (37)$$

To satisfy condition (15) we may write this development as

$$T = 1 + \sum a_n(1-r^n). \quad (38)$$

One may prove that, to satisfy condition (33) and make at the same time the derivatives of equation (30) for $r \to 0$ be satisfied by series (37), all non-zero terms of series (38) must be of the form

$$a_n = a_{4k}, \quad k = 0, 1, 2, 3, \ldots, n. \quad (39)$$

6. THE SOLUTION

Let us estimate the velocity $v(r)$ of the flow. Following equation (31) and introducing the development (38) and condition (39), we get for the velocity

$$v = -\sum\frac{2k}{2k-1}b_k(1-r^{2(2k-1)}) \quad (40)$$

where

$$b_k = \frac{4ka_{4k}}{\phi\,Br}.$$

Following the velocity of flow, we are able to calculate the mass flow rate Q (this being done neglecting the thermal dilatation of the flowing liquid).

The mass flow rate staying constant, the non-dimensional mass flow rate is equal to one

$$Q = 1. \quad (41)$$

This relation, together with equations (33) and (35) give the first three coefficients b_k ($k = 1, 2, 3$) as depending on parameter $\phi' = \phi/4$, which must, at the same time, satisfy equation (34).

One has

$$b_1 = -(3+2\phi') - \tfrac{1}{2}\phi'^2\,Br\left.\frac{d\mu}{dT}\right|_{r=1}$$
$$b_2 = 3(1+\phi') + \phi'^2\,Br\left.\frac{d\mu}{dT}\right|_{r=1}$$
$$b_3 = -(1+\phi') - \tfrac{1}{2}\phi'^2\,Br\left.\frac{d\mu}{dT}\right|_{r=1}. \quad (42)$$

The wall friction per unit surface may be expressed as a function of the parameter ϕ. Equation (29) gives directly the value of the wall friction

$$\mu\,dv/dr.$$

If we introduce $r = 1$ into this equation, we obtain

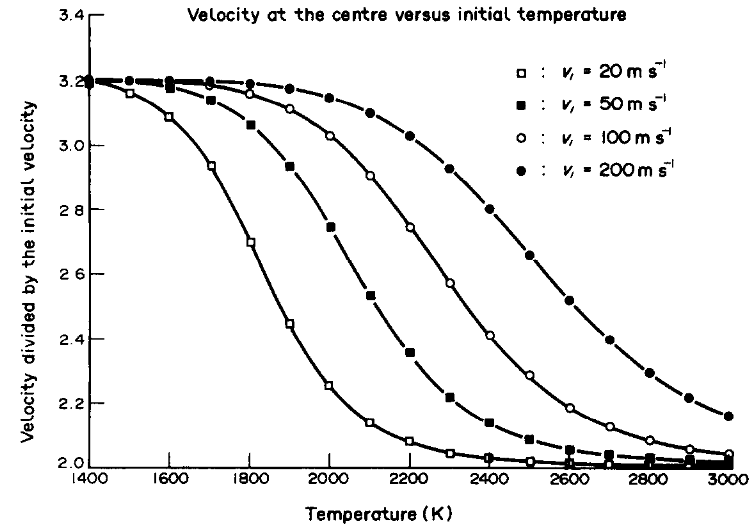

FIG. 2. The velocity at the centre as a function of initial temperature, for different initial velocities.

$$\left(\mu \frac{\mathrm{d}v}{\mathrm{d}r}\right)_{r=1} = \phi. \tag{43}$$

The viscous dissipation per unit volume is

$$D = 2\int_0^1 \mu \left(\frac{\mathrm{d}v}{\mathrm{d}r}\right)^2 r \,\mathrm{d}r. \tag{44}$$

Using equation (23) we obtain

$$D = -\frac{2}{Br}\left[r\frac{\mathrm{d}T}{\mathrm{d}r}\right]_0^1$$

and, with the aid of equations (34) and (32)

$$D = -2\phi. \tag{45}$$

Using this formula, we are now able to calculate the pressure gradient, applying relation (28)

$$\frac{\mathrm{d}p}{\mathrm{d}z} = -\frac{D}{Re\,Eu}. \tag{46}$$

7. DISCUSSION

The temperature in the centre of the tube may now be written (equations (38) and (42)) as

$$T_0 = 1 - \tfrac{1}{6}\left[11 + 5\phi' + \phi'^2\, Br \frac{\mathrm{d}\mu}{\mathrm{d}T}\right]\phi'\, Br.$$

The temperature at the centre must stay finite, even when $Br \to \infty$; in consequence, at the same time there must be $\phi' \to 0$. In the same way, if $\partial\mu/\partial T \to -\infty$ there must also be $\phi' \to 0$. In the case where $\partial\mu/\partial \mathrm{T} \to 0$ (this is the case of constant viscosity as treated by Landau and Lifshitz), we get $\phi' = -1$, as

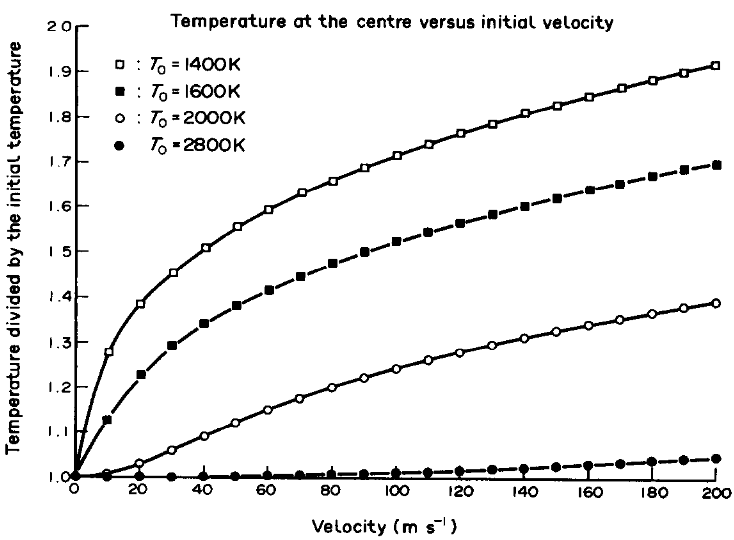

FIG. 3. The temperature at the centre as a function of velocity for different initial temperatures.

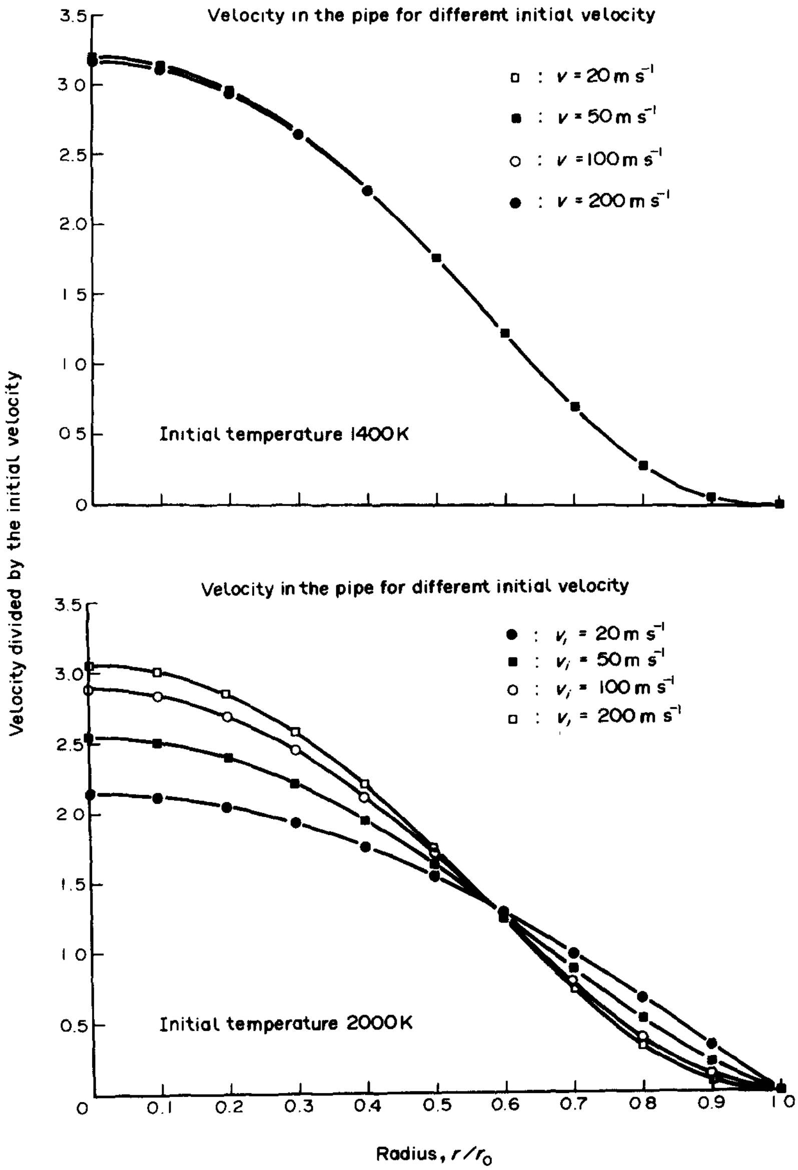

FIG. 4. (a) The velocity distribution for different initial velocities, for an initial temperature of 1400 K. (b) The velocity distribution for different initial velocities, for an initial temperature of 2000 K.

indicated by Landau and Lifshitz. These two limiting cases permit to situate the value of ϕ; there must be

$$-4 \leqslant \phi \leqslant 0.$$

In the case of Landau–Lifshitz (with constant viscosity), the temperature at the centre of the tube is

$$T(0) = 1 + Br.$$

In the case cited by Landau and Lifshitz, the value of *Br* is not limited, which means that the temperature at the centre of the tube is not limited.

In the solution presented here, the value of *Br* is modulated by the value of ϕ and the product $\phi\,Br$ has a limited value; this implies, that the temperature at the centre is always limited.

As to the velocity, it has a maximum value of 2 in the Poiseuille theory. In the present solution this value grows with temperature and varies between 2 and 3.2, as a consequence of the relation

$$v_{\max} = \tfrac{2}{5}\left[8 + 3\phi' - \tfrac{1}{3}\phi'^2\,Br\frac{d\mu}{dT}\right]$$

which, for $Br \to \infty$ (which corresponds to important mass flow rates) must have $\phi' \to 0$, and the value of the velocity of flow at the centre is 3.2.

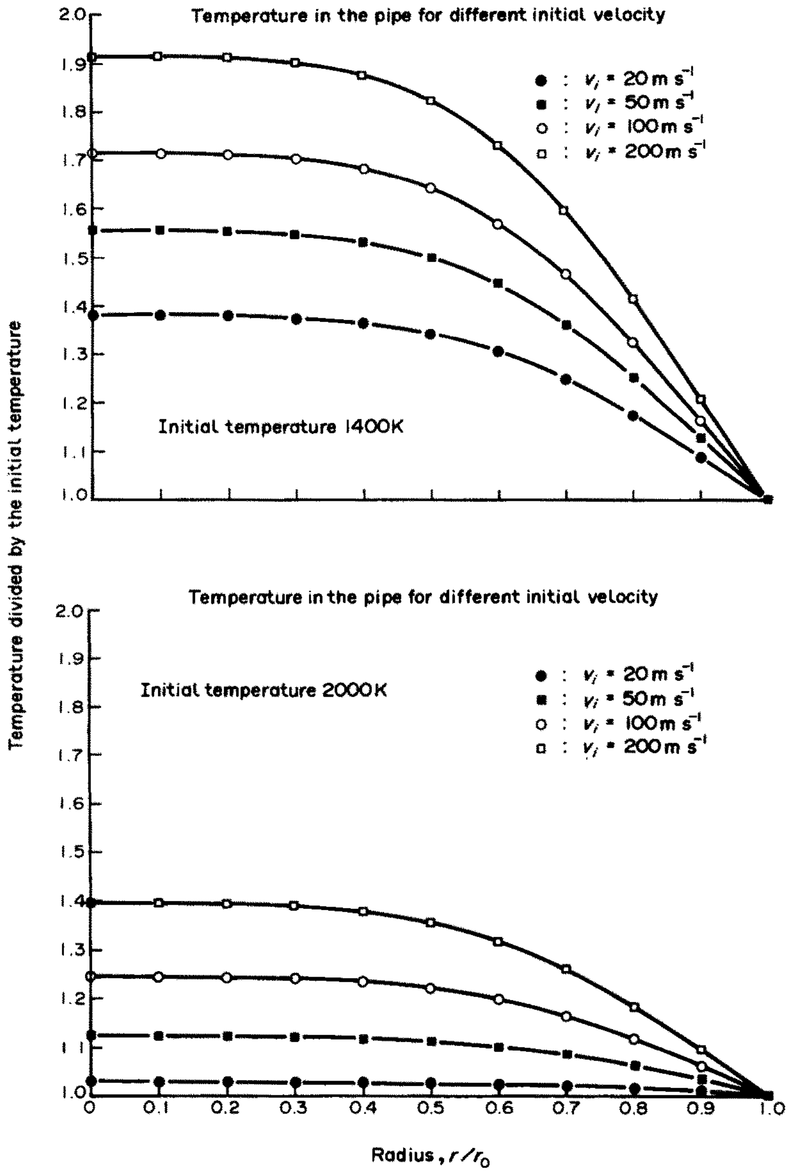

FIG. 5. (a) The temperature distribution for different initial velocities, for an initial temperature of 1400 K. (b) The temperature distribution for different initial velocities, for an initial temperature of 2000 K.

8. NUMERICAL RESULTS

The computations were executed for basaltic magma flows. The characteristics of this magma are as follows [4]:

$$\mu(T) = 1 \times 10^{-6} \exp(26\,170/T)$$

$$\rho = 2700\,\mathrm{kg\,m^{-3}}$$

$$\lambda = 2.93\,\mathrm{W\,m^{-1}\,K^{-1}}.$$

The calculations were made for the wall temperatures

$$T_W = 1400, 1600, 2000, 2800\,\mathrm{K}$$

and for mean velocities of flow

$$v_m = 20, 50, 100, 200\,\mathrm{m\,s^{-1}}.$$

The results are traced on the figures.

Figure 1 shows the viscosity of basaltic magma depending on increasing temperature; this dependence is particularly steep for volcanic magma (as compared with other liquids) and so it is responsible for viscosity effects due to the temperature variations.

Figure 2 shows, that the velocity at the centre of the pipe grows most when the initial temperature is the lowest; this is due to the fact that the friction is more important when the temperature is low. The viscosity diminishes with the internal friction (i.e. with the temperature growth); due to this friction, the temperature in the centre of the pipe (Fig. 3) grows with the velocity. The distribution of the velocity (Fig. 4(a)) across the pipe's diameter depends very little on the mean flow velocity when the temperature is low, but this dependency becomes clear (Fig. 4(b)) for higher initial temperatures. Finally (Figs. 5(a) and (b)) one remarks that the higher the initial temperature, the lower the velocity caused temperature growth (due to lower friction).

9. CONCLUSION

A theory of flow of a liquid, the viscosity and temperature of which are variable, was developed and applied to the basaltic magma flow.

The results show, that the maximum velocity of the flow tends to be 3.2 times the mean velocity, and that the temperature at the centre, high as it is, is limited.

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SUR UN ECOULEMENT DE MAGMA A VISCOSITE VARIABLE

Résumé—L'écoulement d'un fluide dans un tube a été traité par Poiseuille, et Landau et Lifshitz indiquent une généralisation de la solution de Poiseuille pour un fluide dont la température varie le long de l'axe. Platten et Legros traitent un écoulement à viscosité variable, mais l'expression qu'ils utilisent pour décrire la viscosité n'est pas adaptée pour décrire la viscosité d'un liquide. La solution que nous proposons ici concerne l'écoulement d'un magma liquide, dont la viscosité représente une fonction qui dépend fortement de la température. L'exemple que nous calculons numériquement concerne l'écoulement d'un magma basaltique, dont la viscosité obéit à la loi de viscosité des liquides (Atkins), et dont les caractéristiques peuvent être trouvées dans un article de Hardee.

ÜBER MAGMASTRÖMUNG MIT VERÄNDERLICHER ZÄHIGKEIT

Zusammenfassung—Rohrströmungen von Fluiden wurden von Poiseuille abgehandelt und Landau/Lifschitz geben eine Verallgemeinerung der Poiseuilleschen Lösung an, wobei sich die Fluidtemperatur entlang der Strömungsachse verändert. Platten und Legros behandeln eine Strömung veränderlicher Zähigkeit, aber der von ihnen verwandte Ausdruck für die Viskosität ist nicht geeignet zur Beschreibung der Viskosität einer Flüssigkeit. Die von uns hier vorgeschlagene Lösung betrifft einen Magmastrom, wobei die Zähigkeit stark von der Temperatur abhängt. Das von uns numerisch errechnete Beispiel stellt eine Strömung basaltischen Magmas dar, dem das Viskositätsgesetz für Flüssigkeiten zugrunde liegt und dessen Eigenschaften in einem Artikel von Hardee aufgeführt sind.

О ПОТОКЕ МАГМЫ С ПЕРЕМЕННОЙ ВЯЗКОСТЬЮ

Аннотация—Течение жидкости в трубе исследовано Пуазейлем. Ландау и Лифшиц обобщили решение Пуазейля на жидкость с изменяющейся по сечению температурой. Платтен и Легрос рассматривают течение с переменной вязкостью, но используемое ими выражение не описывает вязкость жидкости. Предлагаемое в настоящей работе решение относится к течению магмы с быстро изменяющейся в зависимости от температуры вязкостью. Численные расчеты проводятся для потока базальтовой магмы, подчиняющейся закону вязкости для жидкостей, его характеристики взяты из статьи Харди.